



G

O

D

O

O

O

O

D

# knights + knaves

Z I G Z A G

# permutations?  
identical letters do not appear together

total # permutations

$$\frac{6!}{2! \cdot 2!} = 180$$

repeated letters

ways to choose or together

$$\frac{6!}{2! \cdot 2!} = 180$$

$$\frac{6!}{2! \cdot 2!} = 2 \cdot \frac{4!}{2!} + 4!$$

# RSA Encryption Problem

Bob's private key  $p=13$   $q=11$   
 $e=5$ ?

$$n=pq$$

$$c = M^e \pmod n \text{ Alice}$$

$$M = c^d \pmod n \text{ Bob } ed = 1 \pmod{(p-1)(q-1)}$$

$$5d = 1 \pmod{(13-1)(11-1)}$$

$$5d = 1 \pmod{120} \text{ Linear Congruence}$$

Solve

$$ax = b \pmod m$$

thus  $\gcd(5, 120) = 5 \neq 1$

Thus the choice of 5 for  $c$  is incorrect  
Choice 2:  $c=7$

$$\gcd(7, 120) = 1 \checkmark$$

$$7d = 1 \pmod{120} \quad d=?$$

$$1 = 9 \cdot 7 + 1 \cdot 120$$

$$103 \cdot 7 - 6 \cdot 120 = 721 - 720 = 1 \checkmark$$

$$d = 103$$

solution only exists

# Truth Tables

P	q	$P \wedge q$	$P \vee q$	$P \oplus q$	$P \rightarrow q$	$P \leftrightarrow q$
1	1	1	1	0	1	1
1	0	0	1	1	0	0
0	1	0	1	1	1	0
0	0	0	0	0	1	1

# Domains

- Natural : 0, 1, 2, 3, ...
- Integers : ..., -3, -2, -1, 0, 1, 2, 3
- Rational :  $-\infty, \infty$
- Real :  $-\infty, \infty$  including  $i$

# Predicat Logic

- order of quantifiers doesn't matter if they are the same type

$$\forall x \forall y (x < y) \equiv \forall y \forall x (x < y)$$

- Negations + DeMorgans

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

can bring in or bring out  $\neg$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$\forall x P(x) \equiv \neg \exists x \neg P(x)$$

$$\exists x P(x) \equiv \neg \forall x \neg P(x)$$

# Propositional Logic Rules

$$P \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

$$\neg \neg p = p$$

$$p \wedge (p \vee q) \equiv p$$

$$p \vee (p \wedge q) \equiv p$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$A \leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$$

$$\neg(p \leftrightarrow q) \equiv p \oplus q$$

$$p \leftrightarrow q \equiv (p \vee \neg q) \wedge (\neg p \vee q)$$

# proofs via guess :

$\forall$  chooses first trying to prove false  
 $\exists$  chooses second trying to prove sentence true

$\exists$  was TRUE

### Choosing Problems

Repetitions	Allowed	Order matters?
No	yes	Yes
Permutation $n!$	$n^k$	Yes
r-combinations $\binom{n}{r}$	$\binom{n+r-1}{r-1}$	No

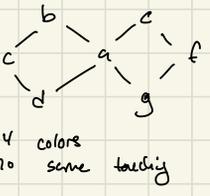
$\binom{n}{r} = \frac{n!}{r!(n-r)!}$   
 $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} + \binom{n}{k+1}$

Objects	Boxes	
	labeled	unlabeled
labeled objects	$n!$	$\binom{n}{k}$
list each permutation	$n!n! \dots n!$	$\binom{n}{k}$
unlabeled	$\binom{n+r-1}{r-1}$	$p(n)$

### APPEASE

#### permutations

$\frac{n!}{a!a!a!} \cdot \frac{s!}{s!} = 2$



- count subsets  
- then count permutations of subsets

$4 \cdot 3^2 \cdot 2^4 = 576$  choices

### Fermat's Little Theorem

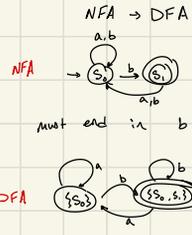
Ex:  $x \equiv 2703 \pmod{5}$   
 $x \equiv 3 \pmod{5}$

$2019 = 4 \cdot 504 + 3$   
 $x \equiv 3 \cdot 504 \cdot 3 \pmod{5}$   
 $x \equiv 1 \cdot 27 \pmod{5}$   
 Thus  $x \equiv 2$

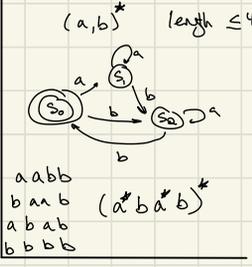
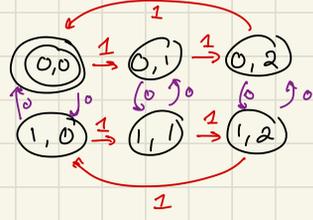
$p=1$   
 $a \equiv 1 \pmod{p}$   
 $3^1 \equiv 1 \pmod{5}$

$2703 \pmod{13} = 12 = -1$

### DFA



even # 0's  
 $\geq$  # 1's



### Proving Big-O

$2^x(x^2+x) = O(2^x)$  (least complexity)  
 $\rightarrow \lim_{x \rightarrow \infty} \frac{2^x(x^2+x)}{12^x} = \lim_{x \rightarrow \infty} (x^2+x) = \infty$   $\rightarrow$  proves  $O(2^x)$  false  
 $\rightarrow \lim_{x \rightarrow \infty} \frac{2^x(x^2+x)}{13^x} = \lim_{x \rightarrow \infty} (\frac{2}{13})^x (x^2+x) = 0$   $\rightarrow$  proves  $O(2^x)$  true

Injective: no two outputs the same (unique)  
 Surjective: every output is satisfied  
 Bijective: both  $\rightarrow$

### Countability

$\mathbb{R}$  not countable  
 $\mathbb{Z}$  not countable  
 $\mathbb{N}$  countable  
 Rational #'s countable  
 $S = \{\text{infinite set}\} \rightarrow$  countable  
 Power set  $(S) =$  uncountable  
 $gcd(a,b) = gcd(a, a-b)$

### Strong Induction

Base case:  $\frac{1}{2} \checkmark, \frac{1}{4} \checkmark$   
 $a_1 = 4, a_2 = 12$   
 prove  $2^n | a_n$   
 $a_n = 10a_{n-1} - 12a_{n-2}$   
 $= 10 \cdot 2^{n-1} - 12 \cdot 2^{n-2}$   
 $2^n (\frac{10}{2} - \frac{12}{4}) = 2^n (5-3) = 2^n \cdot 2$  **proved**

### Weak Induction

prove for all  $N, 5^n - 2^n$  is a multiple of 3  
 ① (base case)  $n=0, 5^0 - 2^0 = 0, 3 \cdot 0 = 0 \checkmark$   
 ② (inductive step)  $5^n + 2^n = 3k$   $\leftarrow$  use for sub  
 $5^{n+1} - 2^{n+1} = 5 \cdot 5^n - 2 \cdot 2^n = 5(5^n - 2^n) + 3 \cdot 2^n$   
 $= 5(3k) + 3 \cdot 2^n = 3(5k + 2^n)$  **IH**

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$   
 det of Fibonacci sequence

## Practice Problems

① write a sentence  $\sqrt{2}$  is irrational

$$\forall n \forall m \left( \sqrt{2} \neq \frac{m}{n} \right)$$

$$\forall n \forall m (n\sqrt{2} \neq m)$$

$$\exists n \exists m (n \neq 0 \wedge n \cdot n \cdot 2 = m \cdot m)$$

②  $f: A \rightarrow B$      $A_1 \subseteq A$  ,  $A_2 \subseteq A$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$f(A_1) = \{ f(a) \in B \mid a \in A_1 \} \quad \text{image of } A_1$$

③  $F_n$      $F_0 = 0$     $F_1 = 1$     $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$   
*def of Fibonacci sequence*

Prove  $F_{mn}$  is a multiple of  $F_n$ .

$n$		0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10		$F_0 = F_{2 \cdot 5}$ multiple of $F_2$
$F_n$		0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,		$F_5$

Proof by induction

Base Case :     $m=0$      $F_{0 \cdot n} = F_0 = 0$     0 is a multiple of all integers ✓

Inductive step :    IH:  $F_{m \cdot n} = l F_n$      $l \in \mathbb{N}$

$$F_{(m+1)n} = F_{mn+n} = F_{mn} F_{n-1} + F_n F_{m+1}$$

IH:  $\rightarrow$

given  $F_{m+k} = F_n F_{m+1} + F_{m+1} F_n$

$$l F_n F_{n-1} + F_n F_{m+1}$$

$$= \frac{F_n (l F_{n-1} + F_{m+1})}{|l| F_n}$$

$m \cdot n = 1$   
 $n = 16$

④  $\{a, b\}$   $b(a|b)^*b$

$bb, bnb, bbb, \dots$

Draw a DFA for language:



⑤ RSA Encryption Problem

Bob's private key  $p=13$   $q=11$

$$n=pq$$

$$c = M^e \pmod n \leftarrow \text{Alice}$$

$$M = c^d \pmod n \leftarrow \text{Bob} \quad ed \equiv 1 \pmod{(p-1)(q-1)}$$

$$5d \equiv 1 \pmod{(13-1)(11-1)}$$

$$5d \equiv 1 \pmod{120} \quad \text{Linear Congruence}$$

Solve

$$ax \equiv b \pmod m \quad \boxed{\gcd(a, m) = 1} \quad \begin{matrix} \swarrow \\ \text{solution} \\ \text{only} \\ \text{exists} \end{matrix}$$

Thus  $\gcd(5, 120) = 5 \neq 1$

Thus the choice of 5 for  $e$  is incorrect

Choice 2:  $e=7$

$$\gcd(7, 120) = 1 \quad \checkmark$$

$$7d \equiv 1 \pmod{120} \quad d = ?$$

$$1 = 5 \cdot 7 + 1 \cdot 120$$

$$103 \cdot 7 - 6 \cdot 120$$

$$= 721 - 720 = 1 \quad \checkmark$$

$$d = 103$$

⑥  $1347$  digit sum = 15  
 $1030$  digit sum = 4

how many 4 digit numbers have a digit sum = 9

$0123$   
 not a 4 digit number

$1000 \rightarrow 999$

9 units into 4 boxes

first box must contain at least 1 digit

only really 8 units into 4 boxes

$1233 \rightarrow 9$

$2412 \rightarrow 9$

$8010 \rightarrow 9$

$$\binom{8+4-1}{4-1} = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165$$

Repetitions	Allowed	order matters?
no	yes	
r-permutations $\frac{n!}{(n-r)!}$	$n^r$	Yes
r-combinations $\binom{n}{r}$	$\binom{n-1}{r-1}$	No

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

# Practice Final Exams

① Allister + Bob

one is knight and one is knave

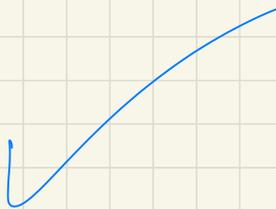
$$t \rightarrow \neg b$$

$$b \rightarrow \neg t$$

bob cannot be

a knight

thus bob is a knave



②  $A = \mathbb{N} \setminus \{0, 1\}$

$$f: A \rightarrow A$$

$$f(15) = f(3 \cdot 5) = 3$$

injective? surjective? bijective?

$$f(2) = 2 \quad \text{not injective}$$

$$f(4) = 2$$

is surjective

③  $a_n = 10a_{n-1} - 12a_{n-2}$  for  $n \geq 2$

$a_1 = 4$   $a_2 = 12$  Prove with strong induction  $2^n \mid a_n$   
for  $n \geq 1$

base case:  $n=1$

$$2^1 = a_1 \Rightarrow$$

$$\frac{4}{2} = 2$$

$$n=2$$

$$4 = 12$$

$$\frac{12}{4} = 3$$

if  $1 \leq m < n$  then  $2^m \mid a_m$

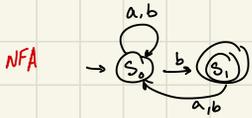
$$a_n = 10a_{n-1} - 12a_{n-2}$$

IH

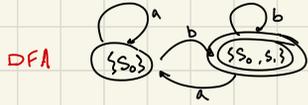
$$= 10 \cdot 2^{n-1} - 12 \cdot 2^{n-2}$$

$$2^n \left( \frac{10}{2} - \frac{12}{4} \right) = 2^n (5 - 3)$$

④ NFA  $\rightarrow$  DFA



must end in b



⑤

~ - -

⑥  $k \geq n$  both positive ints

distribute  $k$  indistinguishable apples into  $n$  distinguishable children

where each child has at least 1 apple

$$\binom{k-n+n-1}{n-1} = \binom{k-1}{n-1}$$

$$\binom{5}{3} = \frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = \boxed{10}$$

in particular for  
 $k=6$   $n=4$

Practice Exam 2

①  $a \rightarrow d \leftrightarrow \neg a$   
 $\neg d \rightarrow \neg a \leftrightarrow \neg a$   
 $\neg d \rightarrow \text{True}$

good at answers + knights

manuscript not in Library

impossible to tell if Arch is truthful

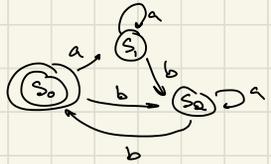
② injective function

$\mathbb{N}^2 \rightarrow \mathbb{Z} \quad |\mathbb{N} \times \mathbb{N}| < \mathbb{Z}$

$f(x,y) = 3^x - 2^y$  not injective

$f_b(x,y) = 3^x \cdot 2^y$

③  $(a,b)^*$  length  $\leq 4$



aabb  
 baab  
 abab  
 bbbb  
 $(a^* b a^* b)^*$

④

$2703^{20112^{15}}$

$1300 \times 2 = 2600$   
 $2600 \times 2 = 5200$   
 $5200 \times 2 = 10400$   
 $10400 \times 2 = 20800$   
 $20800 \times 2 = 41600$   
 $41600 \times 2 = 83200$   
 $83200 \times 2 = 166400$   
 $166400 \times 2 = 332800$   
 $332800 \times 2 = 665600$   
 $665600 \times 2 = 1331200$   
 $1331200 \times 2 = 2662400$   
 $2662400 \times 2 = 5324800$   
 $5324800 \times 2 = 10649600$   
 $10649600 \times 2 = 21299200$   
 $21299200 \times 2 = 42598400$   
 $42598400 \times 2 = 85196800$   
 $85196800 \times 2 = 170393600$   
 $170393600 \times 2 = 340787200$   
 $340787200 \times 2 = 681574400$   
 $681574400 \times 2 = 1363148800$   
 $1363148800 \times 2 = 2726297600$   
 $2726297600 \times 2 = 5452595200$   
 $5452595200 \times 2 = 10905190400$   
 $10905190400 \times 2 = 21810380800$   
 $21810380800 \times 2 = 43620761600$   
 $43620761600 \times 2 = 87241523200$   
 $87241523200 \times 2 = 174483046400$   
 $174483046400 \times 2 = 348966092800$   
 $348966092800 \times 2 = 697932185600$   
 $697932185600 \times 2 = 1395864371200$   
 $1395864371200 \times 2 = 2791728742400$   
 $2791728742400 \times 2 = 5583457484800$   
 $5583457484800 \times 2 = 11166914969600$   
 $11166914969600 \times 2 = 22333829939200$   
 $22333829939200 \times 2 = 44667659878400$   
 $44667659878400 \times 2 = 89335319756800$   
 $89335319756800 \times 2 = 178670639513600$   
 $178670639513600 \times 2 = 357341279027200$   
 $357341279027200 \times 2 = 714682558054400$   
 $714682558054400 \times 2 = 1429365116108800$   
 $1429365116108800 \times 2 = 2858730232217600$   
 $2858730232217600 \times 2 = 5717460464435200$   
 $5717460464435200 \times 2 = 11434920928870400$   
 $11434920928870400 \times 2 = 22869841857740800$   
 $22869841857740800 \times 2 = 45739683715481600$   
 $45739683715481600 \times 2 = 91479367430963200$   
 $91479367430963200 \times 2 = 182958734861926400$   
 $182958734861926400 \times 2 = 365917469723852800$   
 $365917469723852800 \times 2 = 731834939447705600$   
 $731834939447705600 \times 2 = 1463669878895411200$   
 $1463669878895411200 \times 2 = 2927339757790822400$   
 $2927339757790822400 \times 2 = 5854679515581644800$   
 $5854679515581644800 \times 2 = 11709359031163289600$   
 $11709359031163289600 \times 2 = 23418718062326579200$   
 $23418718062326579200 \times 2 = 46837436124653158400$   
 $46837436124653158400 \times 2 = 93674872249306316800$   
 $93674872249306316800 \times 2 = 187349744498612633600$   
 $187349744498612633600 \times 2 = 374699488997225267200$   
 $374699488997225267200 \times 2 = 749398977994450534400$   
 $749398977994450534400 \times 2 = 1498797955988901068800$   
 $1498797955988901068800 \times 2 = 2997595911977802137600$   
 $2997595911977802137600 \times 2 = 5995191823955604275200$   
 $5995191823955604275200 \times 2 = 11990383647911208550400$   
 $11990383647911208550400 \times 2 = 23980767295822417100800$   
 $23980767295822417100800 \times 2 = 47961534591644834201600$   
 $47961534591644834201600 \times 2 = 95923069183289668403200$   
 $95923069183289668403200 \times 2 = 191846138366579336806400$   
 $191846138366579336806400 \times 2 = 383692276733158673612800$   
 $383692276733158673612800 \times 2 = 767384553466317347225600$   
 $767384553466317347225600 \times 2 = 1534769106932634694451200$   
 $1534769106932634694451200 \times 2 = 3069538213865269388902400$   
 $3069538213865269388902400 \times 2 = 6139076427730538777804800$   
 $6139076427730538777804800 \times 2 = 12278152855461077555609600$   
 $12278152855461077555609600 \times 2 = 24556305710922155111219200$   
 $24556305710922155111219200 \times 2 = 49112611421844310222438400$   
 $49112611421844310222438400 \times 2 = 98225222843688620444876800$   
 $98225222843688620444876800 \times 2 = 196450445687377240889753600$   
 $196450445687377240889753600 \times 2 = 392900891374754481779507200$   
 $392900891374754481779507200 \times 2 = 785801782749508963559014400$   
 $785801782749508963559014400 \times 2 = 1571603565499017927118028800$   
 $1571603565499017927118028800 \times 2 = 3143207130998035854236057600$   
 $3143207130998035854236057600 \times 2 = 6286414261996071708472115200$   
 $6286414261996071708472115200 \times 2 = 12572828523992143416944230400$   
 $12572828523992143416944230400 \times 2 = 25145657047984286833888460800$   
 $25145657047984286833888460800 \times 2 = 50291314095968573667776921600$   
 $50291314095968573667776921600 \times 2 = 100582628191937147335553843200$   
 $100582628191937147335553843200 \times 2 = 201165256383874294671107686400$   
 $201165256383874294671107686400 \times 2 = 402330512767748589342215372800$   
 $402330512767748589342215372800 \times 2 = 804661025535497178684430745600$   
 $804661025535497178684430745600 \times 2 = 1609322051070994357368861491200$   
 $1609322051070994357368861491200 \times 2 = 3218644102141988714737722982400$   
 $3218644102141988714737722982400 \times 2 = 6437288204283977429475445964800$   
 $6437288204283977429475445964800 \times 2 = 12874576408567954858950891929600$   
 $12874576408567954858950891929600 \times 2 = 25749152817135909717901783859200$   
 $25749152817135909717901783859200 \times 2 = 51498305634271819435803567718400$   
 $51498305634271819435803567718400 \times 2 = 102996611268543638871607135436800$   
 $102996611268543638871607135436800 \times 2 = 205993222537087277743214270873600$   
 $205993222537087277743214270873600 \times 2 = 411986445074174555486428541747200$   
 $411986445074174555486428541747200 \times 2 = 823972890148349110972857083494400$   
 $823972890148349110972857083494400 \times 2 = 1647945780296698221945714166988800$   
 $1647945780296698221945714166988800 \times 2 = 3295891560593396443891428333977600$   
 $3295891560593396443891428333977600 \times 2 = 6591783121186792887782856667955200$   
 $6591783121186792887782856667955200 \times 2 = 13183566242373585775565713335910400$   
 $13183566242373585775565713335910400 \times 2 = 26367132484747171551131426671820800$   
 $26367132484747171551131426671820800 \times 2 = 52734264969494343102262853343641600$   
 $52734264969494343102262853343641600 \times 2 = 105468529938988686204525706687283200$   
 $105468529938988686204525706687283200 \times 2 = 210937059877977372409051413374566400$   
 $210937059877977372409051413374566400 \times 2 = 421874119755954744818102826749132800$   
 $421874119755954744818102826749132800 \times 2 = 843748239511909489636205653498265600$   
 $843748239511909489636205653498265600 \times 2 = 1687496479023818979272411306996531200$   
 $1687496479023818979272411306996531200 \times 2 = 3374992958047637958544822613993062400$   
 $3374992958047637958544822613993062400 \times 2 = 6749985916095275917089645227986124800$   
 $6749985916095275917089645227986124800 \times 2 = 13499971832190551834179290455972249600$   
 $13499971832190551834179290455972249600 \times 2 = 26999943664381103668358580911944499200$   
 $26999943664381103668358580911944499200 \times 2 = 53999887328762207336717161823888998400$   
 $53999887328762207336717161823888998400 \times 2 = 107999774657524414673434323647777996800$   
 $107999774657524414673434323647777996800 \times 2 = 215999549315048829346868647295555993600$   
 $215999549315048829346868647295555993600 \times 2 = 431999098630097658693737294591111987200$   
 $431999098630097658693737294591111987200 \times 2 = 863998197260195317387474589182223974400$   
 $863998197260195317387474589182223974400 \times 2 = 1727996394520390634774949178364447948800$   
 $1727996394520390634774949178364447948800 \times 2 = 3455992789040781269549898356728895897600$   
 $3455992789040781269549898356728895897600 \times 2 = 6911985578081562539099796713457791795200$   
 $6911985578081562539099796713457791795200 \times 2 = 13823971156163125078199593426915583590400$   
 $13823971156163125078199593426915583590400 \times 2 = 27647942312326250156399186853831167180800$   
 $27647942312326250156399186853831167180800 \times 2 = 55295884624652500312798373707662334361600$   
 $55295884624652500312798373707662334361600 \times 2 = 110591769249305000625596747415324668723200$   
 $110591769249305000625596747415324668723200 \times 2 = 221183538498610001251193494830649337446400$   
 $221183538498610001251193494830649337446400 \times 2 = 442367076997220002502386989661298674892800$   
 $442367076997220002502386989661298674892800 \times 2 = 884734153994440005004773979322597349785600$   
 $884734153994440005004773979322597349785600 \times 2 = 1769468307988880010009547958645194699571200$   
 $1769468307988880010009547958645194699571200 \times 2 = 3538936615977760020019095917290389399142400$   
 $3538936615977760020019095917290389399142400 \times 2 = 7077873231955520040038191834580778798284800$   
 $7077873231955520040038191834580778798284800 \times 2 = 14155746463911040080076383669161557596569600$   
 $14155746463911040080076383669161557596569600 \times 2 = 28311492927822080160152767338323115193139200$   
 $28311492927822080160152767338323115193139200 \times 2 = 56622985855644160320305534676646230386278400$   
 $56622985855644160320305534676646230386278400 \times 2 = 113245971711288320640611069353292460772556800$   
 $113245971711288320640611069353292460772556800 \times 2 = 226491943422576641281222138706584921545113600$   
 $226491943422576641281222138706584921545113600 \times 2 = 452983886845153282562444277413169843090227200$   
 $452983886845153282562444277413169843090227200 \times 2 = 905967773690306565124888554826339686180454400$   
 $905967773690306565124888554826339686180454400 \times 2 = 1811935547380613130249777109652679372360908800$   
 $1811935547380613130249777109652679372360908800 \times 2 = 3623871094761226260499554219305358744721817600$   
 $3623871094761226260499554219305358744721817600 \times 2 = 7247742189522452520999108438610717489443635200$   
 $7247742189522452520999108438610717489443635200 \times 2 = 14495484379044905041998216877221434978887270400$   
 $14495484379044905041998216877221434978887270400 \times 2 = 28990968758089810083996433754442869957774540800$   
 $28990968758089810083996433754442869957774540800 \times 2 = 57981937516179620167992867508885739915549081600$   
 $57981937516179620167992867508885739915549081600 \times 2 = 115963875032359240335985735017771479231098163200$   
 $115963875032359240335985735017771479231098163200 \times 2 = 231927750064718480671971470035542958462196326400$   
 $231927750064718480671971470035542958462196326400 \times 2 = 463855500129436961343942940071085916924392652800$   
 $463855500129436961343942940071085916924392652800 \times 2 = 927711000258873922687885880142171833848785305600$   
 $927711000258873922687885880142171833848785305600 \times 2 = 1855422000517747845375771760284343667697570611200$   
 $1855422000517747845375771760284343667697570611200 \times 2 = 3710844001035495690751543520568687335395141222400$   
 $3710844001035495690751543520568687335395141222400 \times 2 = 7421688002070991381503087041137374670790282444800$   
 $7421688002070991381503087041137374670790282444800 \times 2 = 14843376004141982763006174082274749341580564889600$   
 $14843376004141982763006174082274749341580564889600 \times 2 = 29686752008283965526012348164549498683161129779200$   
 $29686752008283965526012348164549498683161129779200 \times 2 = 59373504016567931052024696329098997366322259558400$   
 $59373504016567931052024696329098997366322259558400 \times 2 = 118747008033135862104049392658197994732644519116800$   
 $118747008033135862104049392658197994732644519116800 \times 2 = 237494016066271724208098785316395989465289038233600$   
 $237494016066271724208098785316395989465289038233600 \times 2 = 474988032132543448416197570632791978930578076467200$   
 $474988032132543448416197570632791978930578076467200 \times 2 = 949976064265086896832395141265583957861156152934400$   
 $949976064265086896832395141265583957861156152934400 \times 2 = 1899952128530173793664790282531167915722312305868800$   
 $1899952128530173793664790282531167915722312305868800 \times 2 = 3799904257060347587329580565062335831444624611737600$   
 $3799904257060347587329580565062335831444624611737600 \times 2 = 759980851412069517465916113012467166288924$

5) How many 10 digit #'s  
not starting with zero

# permutations with 10 digits =  $10!$

# permutations start with zero =  $9!$

answer  $10! - 9! = 3265920$

6) Z I G Z A G

# permutations?

identical letters do not appear together

total # permutations

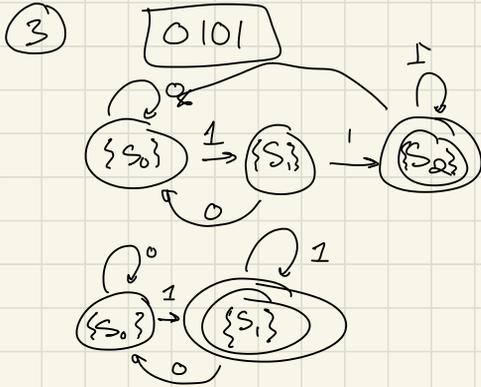
$$\frac{6!}{2!2!} = 150$$

↑ ↑  
repeated  
letters

5! ways to choose Z's  
↑  
repeated  
G's

① All make the same claim must be on the same team

② The union of finite sets is not finite



④  $p = 5$        $q = 3$       odd prime #  
 $m = 3$        $m = 3^2 = 9$

⑤ 9 performances  
3 different operas for each

Amneris	Verdi	Aida
Brunkhilde	Wagners	walkers
Carmen	Bizets	Carmen

(5)

$$\frac{6!}{2!2!2!}$$

(6)

3 dice 18 total options

2 must show same value

$$\frac{r+n-1}{n-1} = \binom{4}{1} \frac{4!}{3!}$$

order doesn't matter 6 ways

order does matter 16 ways

3 rolls produce consecutive  
increasing values

2, 3, 4

order matters ...

$\left. \begin{array}{l} 123 \\ 234 \\ 345 \\ 456 \end{array} \right\} 3!$

- ①
- $a \rightarrow b$
  - $b \rightarrow \neg c$
  - $c \rightarrow d \wedge e$
  - $d \rightarrow \neg a$
  - $e \rightarrow c \wedge \neg e$

- c nave
- b knight
- a knight
- d nave
- c nave

②

$x^y, e^{(\log x)^2}, x^{\sqrt{x}}, e^x$

✓

$2 \log x$

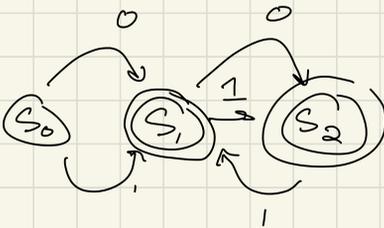
$e^z + e^{\log x}$

$e^z + x \quad e^x \quad x^{x^{\frac{1}{2}}}$

2.5/3

- ③ accepts all words  $\{0,1\}^*$  that do not contain 000

3.5/4



$$\begin{array}{r} 1930 \\ 193 \\ \hline \end{array}$$

④

$$2021^1 \pmod{193}$$

$$2703 \pmod{193}$$

$$\begin{array}{r} 193 \times 10 \\ 1930 \\ \hline x11 \\ \hline \end{array}$$

$$2021^1 \pmod{193}$$

2

1

$$\textcircled{5} \quad \text{gcd}(a, b) = \text{gcd}(a, a-b)$$

$$\text{gcd}(a, 1) = \underline{1}$$

4/5

1, 2, ...

$$n=0$$

$\textcircled{6}$

①

$a \leftrightarrow t_1 \wedge (\neg a \wedge \neg b)$	$a$ end
$b \leftrightarrow a$	$b$ knows
	$t_1$ not treasure
$c \leftrightarrow t_2 \wedge (\neg c \wedge \neg d)$	both naves
$d \leftrightarrow c$	$t_2$ not treasure
$e \leftrightarrow t_3 \wedge \neg f$	$e$ nave
$f \leftrightarrow t_3 \wedge e$	$t_3$ is treasure
	$f$ knight

② weak induction

$$n \in \mathbb{N} \quad (1+x)^n \geq 1+nx$$

$$x > -1$$

$$(1+x)^n = 1+nx$$

IH

base case  $n=0$

$$(1+x)^0 \geq 1+0x$$

$$1 \geq 1 \quad \checkmark$$

induction

$$(1+x)^{n+1} \geq 1+nx+x$$

$$(1+x)^n (1+x) \geq (1+x) + nx$$

$$(1+nx)(1+x) \geq (1+x) + (nx)$$

true

$$1+nx \geq 1 + \frac{nx}{1+x}$$

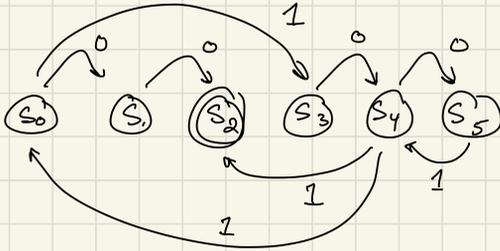
IH

3

even # 0's

# 1's multiple of 3

minimal automaton 6 states



00  
10011

4

smallest > 2022

$$x \equiv 3 \pmod{4}$$

$$x \equiv 3 \pmod{27}$$

$$x \equiv 3 \pmod{25}$$